Probability of Consensus in Spatial Opinion Models with Confidence Threshold

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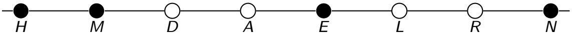
OUTLINE

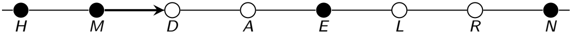
Basic Voter Model

Markov chain $\eta_t : \mathbb{Z} \to \{0, 1\}$



All opinions are equally likely

Each individual mimics a randomly chosen neighbor at rate one



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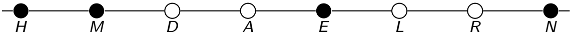

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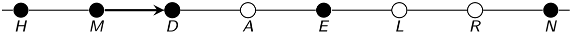
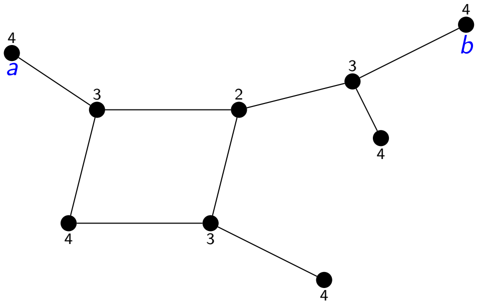


Graph Theory – tools

$G = (V, E)$



The (geodesic) distance from a to b, $d(a, b) = 4$

Graph Theory – tools

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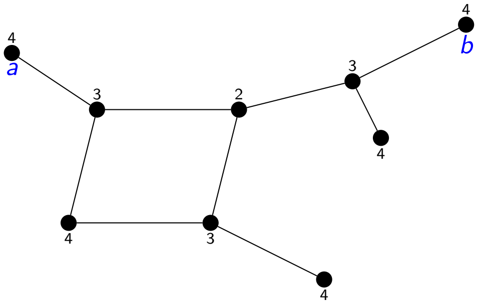

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eccentricity ϵ of a vertex v

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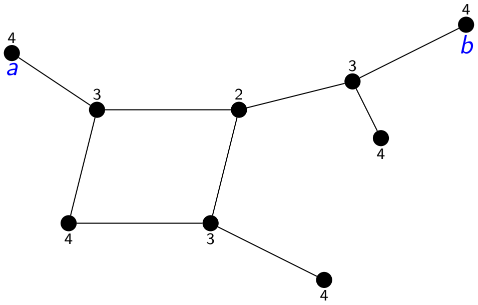

The (geodesic) distance from a to b, $d(a, b) = 4$

eccentricity ϵ of a vertex v

radius $\mathbf{r} = 2$, diameter $\mathbf{d} = 4$

Graph Theory – example

radius $\mathbf{r} = 1$, diameter $\mathbf{d} = 2$

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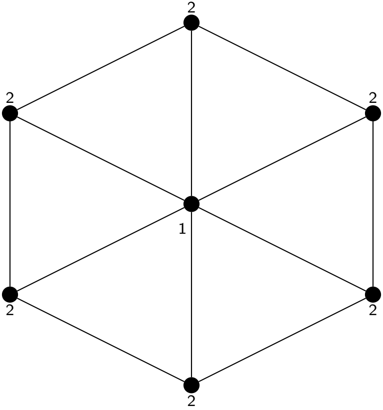

General Opinion Model with Confidence Threshold τ

$\mathcal{G} = \mathbb{Z} = (\mathcal{V}, \mathcal{E})$ is a spatial graph

Markov chain $\xi_t : \mathbb{Z} \to V$,
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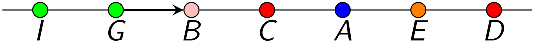
Individuals interact if and only if their opinion distance $d(a, b) \leq \tau$

General Opinion Model – example

Let $\tau = 2$

Interaction of G and B in $\mathcal{G}$



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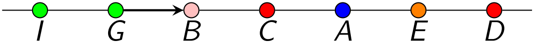
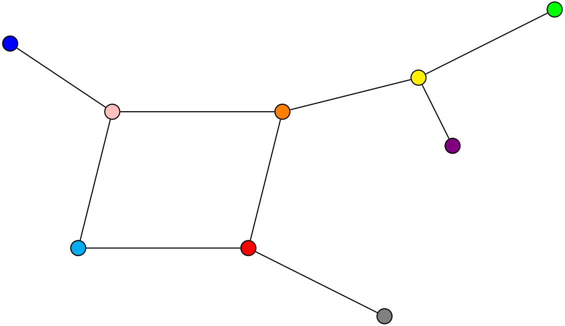


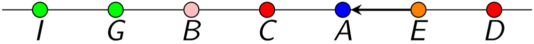
Below is G, the opinion graph
$3 = d(\xi(G), \xi(B)) > \tau \implies$ no interaction

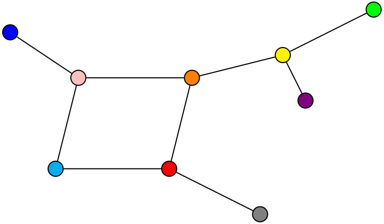


General Opinion Model – example

$\tau = 2$

Interaction of E and A





General Opinion Model – example

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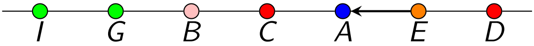
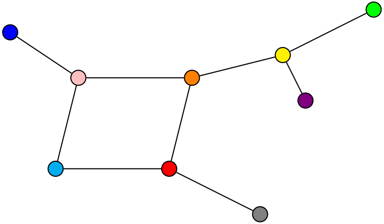
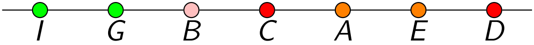




$2 = d(\xi(E), \xi(A)) = \tau \implies$ interaction



Imitation Model

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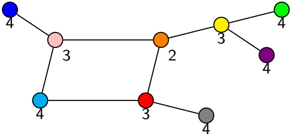
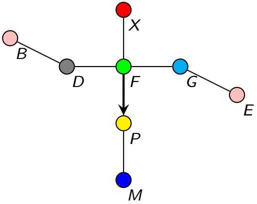
Markov chain $\xi_t : \mathcal{V} \to V$



Each individual imitates a randomly chosen neighbor at rate one

Individuals interact if and only if their opinion distance is at most $\tau\ (= 2)$



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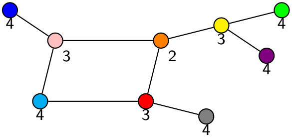

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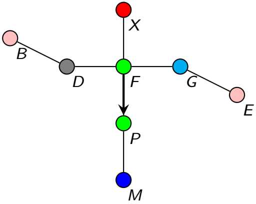

Imitation Model

We define the process

$$X_t = \sum_{x \in \mathcal{V}} \mathbf{1}\{\epsilon(\xi_t(x)) \leq \tau\} = |\{x \in \mathcal{V} : \epsilon(\xi_t(x)) \leq \tau\}|,$$

that keeps track of the number of individuals whose opinion has eccentricity $\epsilon \leq \tau$

Imitation Model – blueprint

Lemma. Time to fixation T is an almost surely finite stopping time

Lemma. (X_t) martingale

$\downarrow$

Optional Stopping Theorem to (X_t)

$\downarrow$

$P(\xi_T \equiv \text{consensus}) > 0$

Imitation Model

$\rightarrow$ $\tau \geq \mathbf{d}$

$$P(\xi_T \equiv \text{consensus}) = 1$$

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→ $\tau \geq \mathbf{d}$

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→ $\tau \in [\mathbf{r}, \mathbf{d})$

$$P(\xi_T \equiv \text{consensus}) \geq \frac{|\{a \in V : \epsilon(a) \leq \tau\}|}{|V|} > 0$$

Attraction Model

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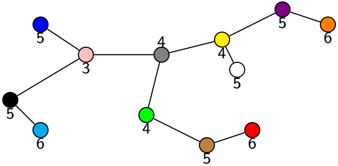
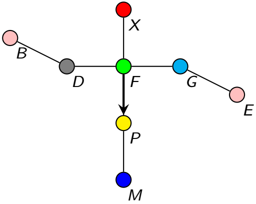
Markov chain $\zeta_t : \mathcal{V} \to V$



Each individual moves one opinion distance closer to a randomly chosen neighbor at rate one

Individuals interact if and only if their opinion distance is at most $\tau\ (= 2)$



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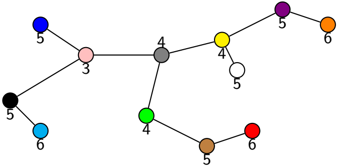

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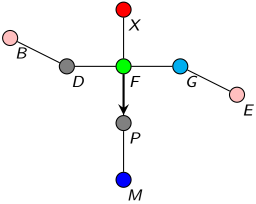

Attraction Model

The opinion graph of our model is acyclic since our result follows

Lemma (eccentricity inequalities)

$\epsilon_{i'} + \epsilon_{j'} \leq \epsilon_i + \epsilon_j$

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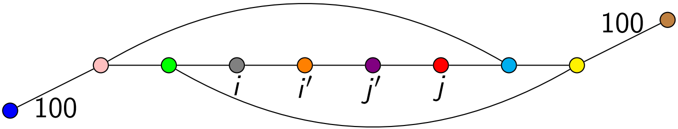
A non-example of a cyclic opinion graph



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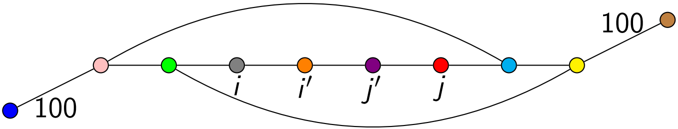

$\epsilon_i = 102 = \epsilon_j; \quad \epsilon_{i'} = 103 = \epsilon_{j'}$

This implies that $\epsilon_{i'} + \epsilon_{j'} \not\leq \epsilon_i + \epsilon_j$

Attraction Model

We define the process

$$(Z_t) = \sum_{x \in \mathcal{V}} (\epsilon(\zeta_t(x)) - \mathbf{r}) = \sum_{a \in V} (\epsilon(a) - \mathbf{r}) \left| \{x \in \mathcal{V} : \zeta_t(x) = a\} \right|,$$

that keeps track of the eccentricity of the individuals' opinions

Attraction Model – blueprint

Eccentricity inequality satisfied

$\downarrow$

Lemma. (Z_t) supermartingale
Lemma. Time to fixation T is an almost surely finite stopping time

$\downarrow$

Optional Stopping Theorem to (Z_t)

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$P(\zeta_T \equiv \text{consensus}) > 0$

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$$P(\zeta_T \equiv \text{consensus}) = 1$$

$\rightarrow \; \tau \in [\mathbf{r}, \mathbf{d})$

$$P(\zeta_T \equiv \text{consensus}) \geq 1 - \frac{1}{|V|} \sum_{a \in V} \left(\frac{\epsilon(a) - \mathbf{r}}{\tau + 1 - \mathbf{r}} \right)$$

Attraction Model – G: full n-ary tree

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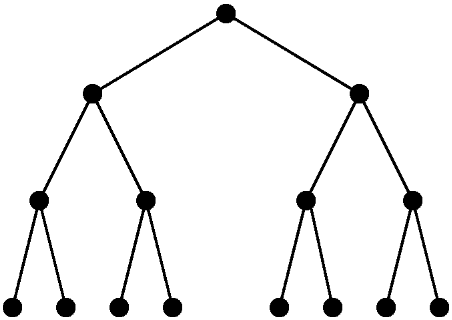

$\tau \in [\mathbf{r}, 2\mathbf{r})$:

$$P(\zeta_T \equiv consensus) \geq 1 - \left(\frac{1}{\tau + 1 - \mathbf{r}}\right)\left(\frac{n(\mathbf{r}n^{\mathbf{r}+1} - (\mathbf{r}+1)n^{\mathbf{r}} + 1)}{(1-n)(1-n^{\mathbf{r}+1})}\right)$$

Attraction Model – G: star-like graph

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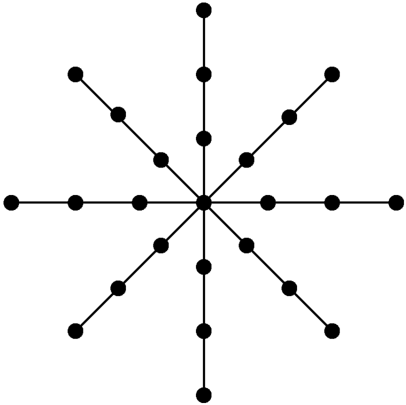

$\tau \in [\mathbf{r}, 2\mathbf{r})$:

$$P(\zeta_T \equiv consensus) \geq 1 - \left(\frac{1}{\tau + 1 - \mathbf{r}}\right)\left(\frac{\mathbf{r}(\mathbf{r}+1)n}{2(1+\mathbf{r}n)}\right)$$

Thank you!