NAVIGATING $PU(2)$ WITH GOLDEN GATES

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FIELD OF DREAMS

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A PRACTICE PROBLEM $U(1)$

$$G = U(1) = \{ z \in \mathbb{C}^* : z\bar{z} = zz^* = 1 \}$$

$$\cong \mathbb{R}/\mathbb{Z} \; ; \quad \theta \to e^{2\pi i \theta}$$

SEEK THE BEST TOPOLOGICAL GENERATOR OF G.

$R_\alpha : \theta \mapsto \theta + \alpha$, ROTATION BY α

$\langle R_\alpha \rangle$ THE GROUP GENERATED BY R_α

$$R_\alpha^j = R_{j\alpha} \quad , \quad j = 1, 2, \ldots, k,$$

$\overline{\langle R_\alpha \rangle} = G$ IFF α IS IRRATIONAL.

HOW WELL DOES R_α^j, $j = 1, \ldots, k$ COVER G?

$$L_k(\alpha) := \max_{\substack{I \subset G \\ I \cap \{\alpha, 2\alpha, \ldots, k\alpha\} = \phi}} |I| \qquad I\text{-INTERVAL}$$

CLEARLY $L_k(\alpha) \geq 1/k$

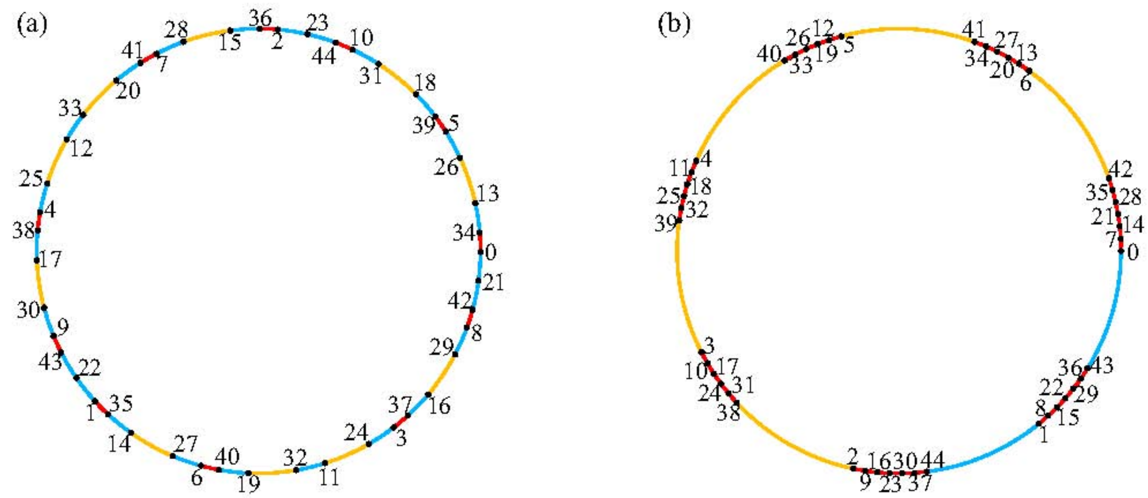


Figure 1. (a) The first 45 iterates of $x = 0$ under R_ϕ for $\phi = \left(\sqrt{5} - 1\right)/2$. (b) The first 45 iterates of $x = 0$ under R_θ for $\theta = 4 - \pi$. Iterates are labelled and arcs between consecutive points in each orbit are colored according to their relative length.

<u>THEOREM</u> (GRAHAM/VAN LINDT, V. SÓS):

$$\overline{\lim_{k \to \infty}} \; k L_\alpha(k) \geq 1 + \frac{2}{\sqrt{5}},$$

WITH EQUALITY IFF $\alpha = \phi = \frac{1+\sqrt{5}}{2}$.

MOREOVER GIVEN $I \subset \mathbb{R}/\mathbb{Z}$ AN INTERVAL DETERMINE IF THERE IS $1 \leq j \leq k$ WITH $j\phi \in I$?

ONE CAN USE EUCLID'S ALGORITHM FOR GCD'S TO ANSWER THIS IN POLYLOG(k) STEPS!

$\Rightarrow$ R_ϕ IS THE OPTIMAL TOPOLOGICAL GENERATOR OF $U(1)$

AND ONE CAN NAVIGATE EFFICIENTLY WITH R_ϕ.

OUR PROBLEM IS TO DO THE SAME FOR $G = SU(2)$ OR $PU(2)$.

$$G = SU(2) = \{ g \in GL_2 : g g^* = I, \det g = 1 \}$$

$$(PU(2) = U(2) / \text{SCALAR MATRICES})$$

G IS A TOPOLOGICAL (COMPACT) GROUP WITH BI-INVARIANT METRIC

$$d_G^2(g,h) = 1 - \frac{|\mathrm{trace}(g^* h)|}{2}$$

$$d_G(gy, hy) = d_G(yg, yh) = d_G(g,h)$$

$$g, h, y \in G.$$

VOL_G IS THE CORRESPONDING INVARIANT HAAR MEASURE ON G

$$VOL(G) = 1, \quad VOL(Ag) = Vol(gA) = Vol(A).$$

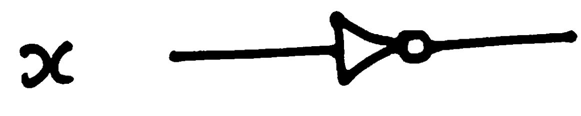
- OUR AIM IS TO GIVE OPTIMAL TOPOLOGICAL GENERATORS OF G AND TO NAVIGATE EFFICIENTLY.

CLASSICAL COMPUTING CIRCUIT MODEL

SINGLE BIT $x \in \{0,1\}$

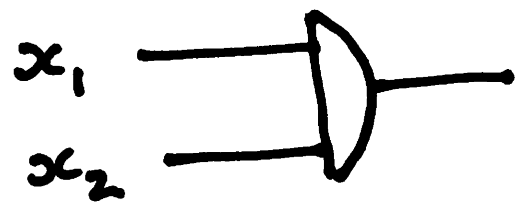
- ONE BIT NOT GATE

$\sim x$,



- TWO BIT AND GATE

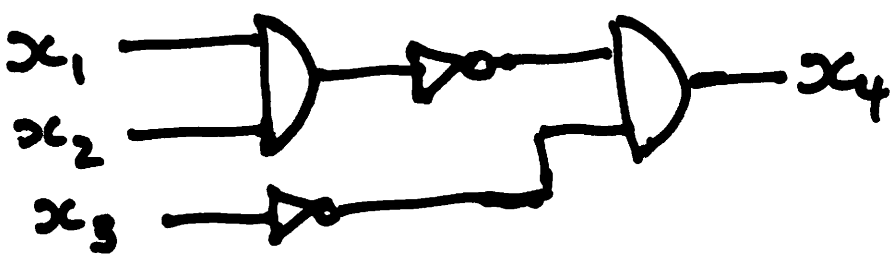
$x_1 \wedge x_2$,



AN n-BIT CIRCUIT IS A BOOLEAN FUNCTION

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

EG:



, $x_4 = \sim(x_1 \wedge x_2) \wedge \sim x_3$

THE GATES $\{$NOT, AND$\}$ ARE UNIVERSAL; EVERY f CAN BE EXPRESSED AS A CIRCUIT USING THESE GATES.

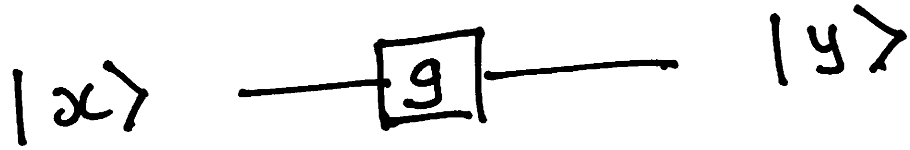
- THE SIZE OF A CIRCUIT IS ITS COMPLEXITY.

THEORETICAL QUANTUM COMPUTING

A SINGLE QUBIT STATE IS A UNIT VECTOR ψ IN $\mathbb{C}^2$

$$\psi = (\psi_1, \psi_2), \quad |\psi|^2 = \psi_1 \overline{\psi_1} + \psi_2 \overline{\psi_2} = 1$$

- A ONE BIT QUANTUM GATE IS AN ELEMENT $g \in U(2)$ (OR $SU(2)$, $PU(2) := G$) ACTING ON ψ'S



$U(2)$ IS THE GROUP OF 2×2 UNITARY MATRICES

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad g^* = \begin{bmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{bmatrix}; \quad g g^* = I$$

$$SU(2): \quad g = \begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

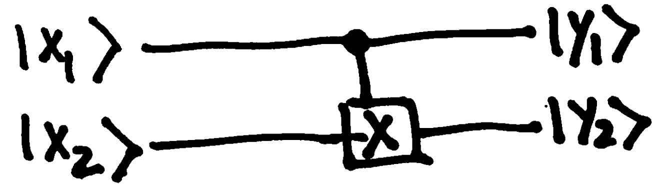
n-QUBITS ARE VECTORS IN $(\mathbb{C}^2)^{\otimes n}$

VECTOR SPACE OF DIMENSION 2^n

- TWO BIT QUANTUM GATE

XOR (OR CNOT) ON BASIS $e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1$

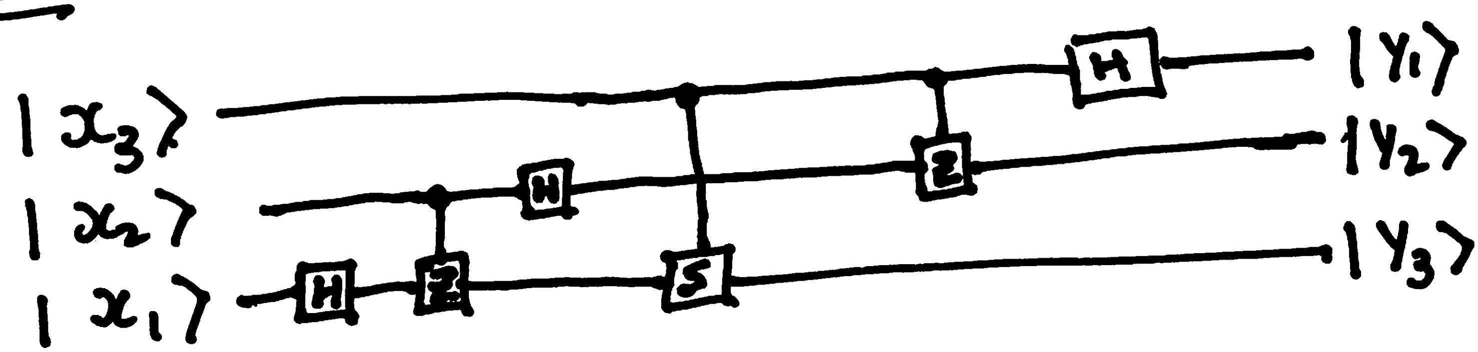
$$XOR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix};$$



$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

THE ONE BIT GATES $g \in G$, TOGETHER WITH XOR ARE UNIVERSAL FOR QUANTUM COMPUTING. THAT IS ANY $g \in U(2^n)$ CAN BE EXPRESSED AS A CIRCUIT IN THESE.

EG: THREE BIT QUANTUM FOURIER TRANSFORM



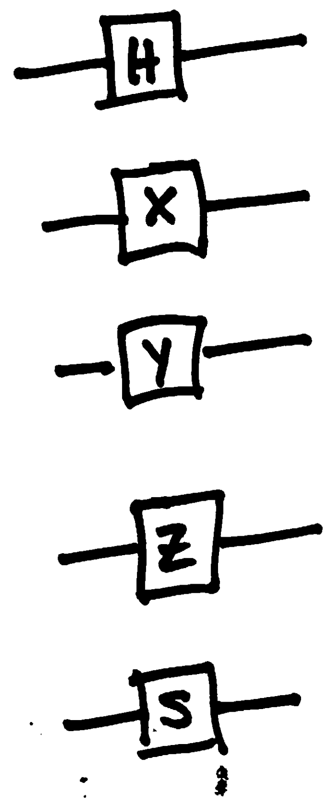
HADAMARD $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

PAULI $\quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

PAULI $\quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

PAULI $\quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

PHASE $\quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$



THESE ELEMENT GENERATE THE CLIFFORD GROUP C_{24} OF ORDER 24 IN G.

C_{24} IS NOT DENSE IN G.

MOST TREATMENTS ADD THE "T-GATE"

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \quad \text{"}\frac{\pi}{8}\text{-GATE"}$$

C_{24} PLUS T GENERATE A DENSE SUBGROUP AND ARE AN EXAMPLE OF A GOLDEN GATE SET (KLIUCHNIKOV-MASLOV-MOSCA).

$F = \{C_{24}, T, XOR\}$ IS UNIVERSAL AND HAS SOME OPTIMAL PROPERTIES.

- THE T-GATE IS CONSIDERED EXPENSIVE IN CIRCUITS IN G FROM VARIOUS POINTS OF VIEW INCLUDING FAULT TOLERANCE.

$\Rightarrow$ THE COMPLEXITY OF A CIRCUIT IN $C_{24} + T$ IS THE T-COUNT, IE NUMBER OF APPLICATIONS OF T.

SU(2) DOUBLE COVERS SO(3)

$g \in SU(2)$, $g = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$, TRACE(g) = 0 $\Longleftrightarrow$

$$g = \begin{bmatrix} ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & -ix_2 \end{bmatrix}$$

$(x_2, x_3, x_4) \longleftrightarrow$ trace(g) = 0

$x_2^2 + x_3^2 + x_4^2 = 1$

$$(x_2, x_3, x_4) \longrightarrow g^* \begin{bmatrix} ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & -ix_2 \end{bmatrix} g$$

gives a rotation in (x_2, x_3, x_4), call it $\pi(g)$. $\pi(g) \in SO(3)$

$$SU(2) \xrightarrow{\pi} SO(3).$$

$C_{24} \rightarrow$ ROTATIONS OF A CUBE.

SOLOVAY-KITAEV THEOREM:

GIVEN A, B TOPOLOGICAL GENERATORS OF G, FOR $\varepsilon > 0$ AND $g \in G$ ONE CAN FIND A WORD W(A, B) OF LENGTH $O((\log 1/\varepsilon)^c)$ AND IN AS MANY STEPS S.T. $d(W, g) < \varepsilon$ (HERE $c \approx 4$).

THIS GIVES A CRUDE BUT REASONABLY EFFICIENT ALGORITHM TO NAVIGATE G.

BASIC PROBLEM; OPTIMAL GENERATORS FOR G:

GIVEN A FINITE SUBGROUP C OF G TO FIND AN INVOLUTION T $(T^2 = 1)$ SUCH THAT $F = C \cup \{T\}$ GENERATES G TOPOLOGICALLY OPTIMALLY IN TERM OF COVERING G WITH SMALL T-COUNT, AND WITH AN EFFICIENT NAVIGATION ALGORITHM.

THE CIRCUITS $S_F(t)$ IN THE GATES F WITH T-COUNT t ARE OF THE FORM

$$c_1 T c_2 T \dots c_t T \quad , \; c_j \in C$$

$$|S_F(t)| = |C|^2 (|C|-1)^{t-1} ; t \geq 1$$

THE PROPERTIES THAT WE WANT ARE

(I) $S_F(t)$, $t \le k$ ARE DISTINCT ELEMENTS IN G.

(II). IF $N_F(k) = \left| \bigcup_{t \le k} S_F(t) \right|$, THEN THESE $N(k)$ POINTS SHOULD COVER G ESSENTIALLY OPTIMALLY. IF B IS A BALL CENTERED AT $I \in G$ THEN

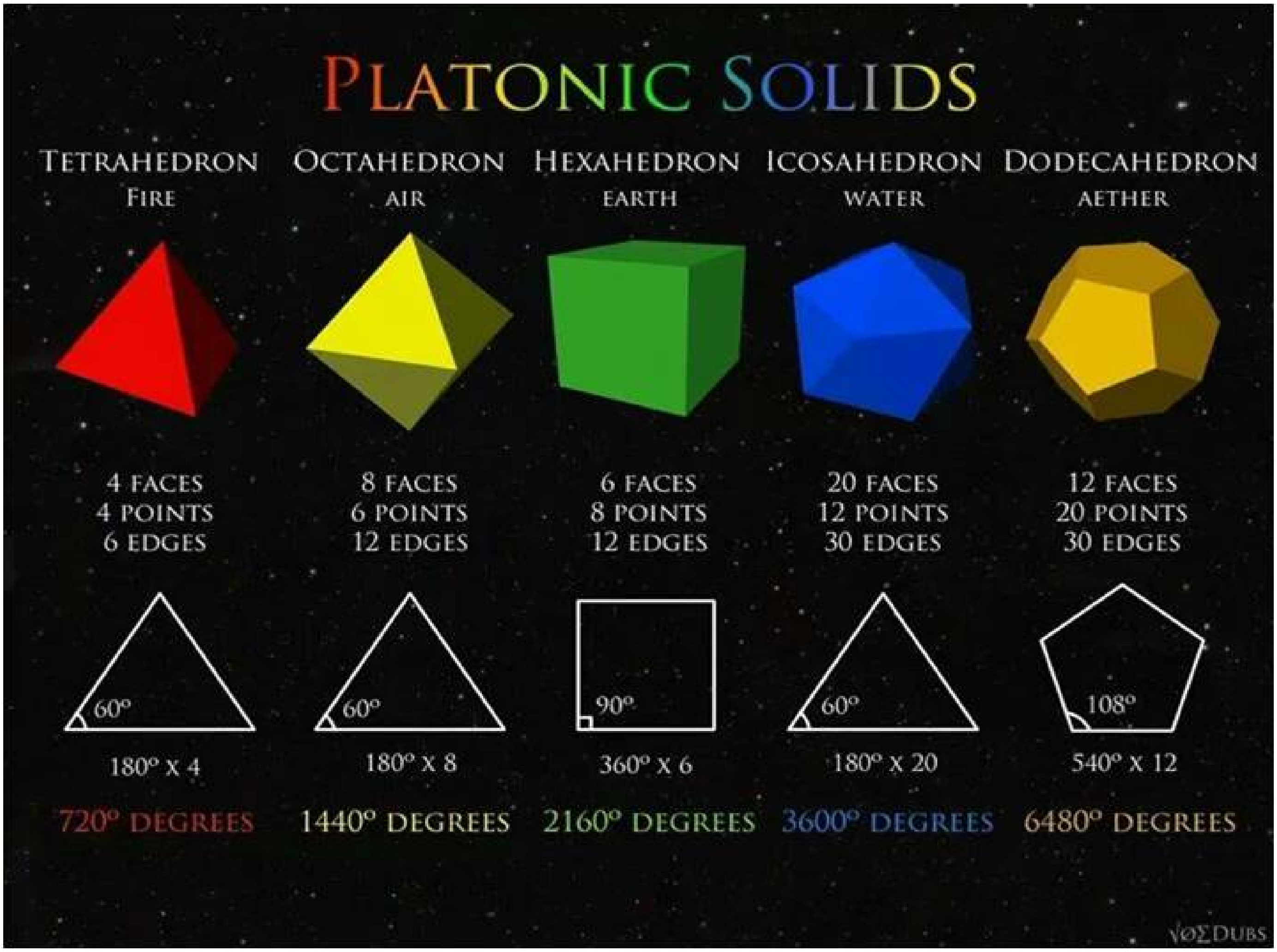
$$\bigcup_{t \le k} \bigcup_{g \in S_F(t)} Bg \quad \text{COVERS } G.$$

FOR THIS TO HAPPEN WE NEED

$$Vol(B)\, N_F(k) \geq 1 .$$

WE RELAX THIS A LITTLE, REQUIRING THAT IF $Vol(B)\, N_F(k) \longrightarrow \infty$ VERY SLOWLY THEN WE (ALMOST) COVER G.

(III) NAVIGATION: GIVEN $x \in G$ AND A BALL B CENTERED AT x, FIND EFFICIENTLY (IE IN POLY k) A $g \in \left[\bigcup_{t \le k} S_F(t) \right] \cap B$, IF SUCH EXISTS.


PLATONIC SOLIDS
TETRAHEDRON
FIRE
OCTAHEDRON
AIR
HEXAHEDRON
EARTH
ICOSAHEDRON
WATER
DODECAHEDRON
AETHER
4 FACES
4 POINTS
6 EDGES
8 FACES
6 POINTS
12 EDGES
6 FACES
8 POINTS
12 EDGES
20 FACES
12 POINTS
30 EDGES
12 FACES
20 POINTS
30 EDGES
60°
60°
90°
60°
108°
180° X 4
180° X 8
360° X 6
180° X 20
540° X 12
720° DEGREES
1440° DEGREES
2160° DEGREES
3600° DEGREES
6480° DEGREES
√ØΣDUBS

THE (INTERESTING) FINITE SUBGROUPS OF G ARISE AS THE ROTATIONAL SYMMETRIES OF THE 5 PLATONIC SOLIDS.

TETRHEDRON	,	A_4	$\lvert A_4\rvert = 12$
CUBE / OCTAHEDRON	,	S_4	$\lvert S_4\rvert = 24$
DODECAHEDRON/ ICOSAHEDRON.	,	A_5	$\lvert A_5\rvert = 60$.

SUPER-GOLDEN GATES (PARZANCHEVSKI-S):

(1) CUBE, PAULI GROUP.

$$C_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \quad T_4 = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

(2) MINIMAL CLIFFORD (OCTAHEDRON).

$$C_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right\}, \quad T_3 = \begin{pmatrix} 0 & \sqrt{2} \\ 1+i & 0 \end{pmatrix}$$

(3) TETRAHEDRON, HURWITZ

$$C_{12} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right\rangle, \quad T_{12} = \begin{pmatrix} 3 & 1-i \\ 1+i & -3 \end{pmatrix}$$

4) OCTAHEDRON, CLIFFORD

$$C_{24} = \langle S, H \rangle \quad , \quad T_{24} = \begin{pmatrix} -1-\sqrt{2} & 2-\sqrt{2}+i \\ 2-\sqrt{2}-i & 1+\sqrt{2} \end{pmatrix}$$

5) ICOSAHEDRON, KLEIN GROUP.

$$C_{60} = \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & \phi - i/\phi \\ \phi + i/\phi & -1 \end{pmatrix} \right\rangle$$

$$\phi = \frac{1+\sqrt{5}}{2} \text{ (GOLDEN RATIO)}, \quad T_{60} = \begin{pmatrix} 2+\phi & 1-i \\ 1+i & -2-\phi \end{pmatrix}$$

THEOREM:

THESE SUPER GATE SETS SATISFY (I), (II) AND PART OF (III).

MORE PRECISELY CONCERNING NAVIGATION (III)

IF $g \in G$ IS DIAGONAL AND ONE CAN FACTOR INTEGERS EFFICIENTLY, THEN THERE IS A HEURISTIC EFFICIENT ALGORITHM (ROSS-SELINGER) WHICH FINDS THE SHORTEST CIRCUIT WITH $k \leq K$ BEST APPROXIMATING g. ON THE OTHER HAND IF g IS A GENERAL ELEMENT IN G THEN FINDING THE SHORTEST CIRCUIT APPROXIMATING g IS ESSENTIALLY NP-COMPLETE!

NEVER-THE-LESS A CIRCUIT 3-TIMES LONGER THAN THE SHORTEST ONE CAN BE FOUND EFFICIENTLY.

SOME INGREDIENTS IN THE ANALYSIS:

WE SAW THAT

$$SU(2) \xleftrightarrow{\text{ISOMETRIC}} S^3 \subset \mathbb{R}^4$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

THE ARITHMETIC SET UP FOR THESE GOLDEN GATES IS SO THAT THE WORDS IN F OF T-COUNT t CORRESPOND TO SOLUTIONS IN INTEGERS TO

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = p^t \qquad (*)$$

HERE $p = 3$ FOR C_4

$p = 11$ FOR C_{12}

FOR C_{24} $(*)$ IS TO BE SOLVED IN INTEGERS IN $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ AND $\cancel{\mathcal{P} = \sqrt{2}}$; NORM$(\mathcal{P}) = 23$

$\mathcal{P} \in \mathcal{O}$

FOR C_{60} $(*)$ IS TO BE SOLVED IN $\mathcal{O}$ THE INTEGERS IN $\mathbb{Q}(\sqrt{5})$, $\mathcal{P}$ IS IN $\mathcal{O}$

$N(\mathcal{P}) = 59$.

PROBLEM (II) BECOMES ONE OF VERY STRONG APPROXIMATION FOR

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$$

LET THE INTEGER SOLUTIONS BE $S(n)$, $|S(n)| = N(n)$ $(\approx n)$

PROJECT THESE $N(n)$ POINTS ONTO S^3

$$x \to \frac{x}{\sqrt{n}}, \quad x \in S(n).$$

HOW WELL DO THESE $N(n)$ POINTS COVER S^3 ?

OPTIMALLY IN THE SENSE OF (II) !

RELIES ON THE RAMANUJAN CONJECTURES $=$ DELIGNE'S THEOREM.

FOR THE NAVIGATION WE NEED TO FIND SOLUTIONS TO SUMS OF SQUARES

$$x_1^2 + x_2^2 = n \qquad (1)$$

IT IS SOLVABLE IFF $n = p_1^{e_1} \cdots p_k^{e_k}$ WITH e_j EVEN WHEN $p_j \equiv 3 \ (4)$.

CAN WE FIND A SOLUTION EFFICIENTLY, IE IN POLY$(\log n)$ STEPS?

- FOR $p \equiv 1 \ (4)$ A PRIME SCHOOF GIVES A $(\log p)^9$ ALGORITHM TO FIND x_1 AND x_2.

HENCE IF WE CAN FACTOR n EFFICIENTLY WE CAN SOLVE (1) EFFICIENTLY BY SIMPLY MULTIPYING THE SOLUTIONS IN $\mathbb{Z}[\sqrt{-1}]$.

<u>NOTE:</u> WHILE FACTORING IS NOT KNOWN TO BE EFFICIENT (I.E. IN $\mathcal{P}$) THERE IS NO THEORETICAL EVIDENCE THAT IT IS NOT IN $\mathcal{P}$. A QUANTUM COMPUTER CAN FACTOR EFFICIENTLY (SHOR'S THEOREM) SO WE MIGHT WANT TO AVOID FACTORING IN BUILDING EFFICIENT GATES. THE ROSS-SELINGER ALGORITHM FOR NAVIGATING TO DIAGONAL $\mathfrak{Z} \in \mathbb{G}$ WILL YIELD A SOLUTION WHICH HAS A $(1+o(1))$ TIMES LONGER T-COUNT THAN THE OPTIMAL, WITHOUT APPEALING TO FACTORING.

IF WE ADD TO THE QUADRATIC DIOPHANTINE PROBLEM (1) A SIMPLE APPROXIMATION CONDITION THINGS CHANGE DRAMATICALLY.

- THE TASK: GIVEN $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{Q}$ FIND INTEGERS x_1, x_2 S.T.

$$\left.\begin{aligned} x_1^2 + x_2^2 &= n \\ \alpha \le x_1/x_2 &\le \beta \end{aligned}\right\}$$

IS NP-COMPLETE!

IDEA OF PROOF: REDUCE TO SUBSUM PROBLEM GIVEN $t_1, \dots, t_m, \ell$ INTEGERS IS THERE $\varepsilon_1, \dots, \varepsilon_m$, $\varepsilon_j = 0, 1$ S.T.

$$\varepsilon_1 t_1 + \dots + \varepsilon_m t_m = \ell .$$

EXPLOIT n's OF THE FORM $p_1 p_2 \cdots p_m$ p_j's SMALL.

THE MOST DIFFICULT PART OF THE NAVIGATION ALGORITHM IS TO SOLVE:

TASK: GIVEN $n \in \mathbb{N}$, $\xi \in S^3$
AND A BALL B CENTERED AT ξ,
FIND $x \in S(n)$ (IF SUCH EXISTS)
SUCH THAT $\tilde{x} = \frac{x}{\sqrt{n}} \in B$.

THE TASK IS NP-COMPLETE, BUT IF $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ HAS TWO OF ITS CO-ORDINATES EQUAL TO 0 ("DIAGONAL") THEN ASSUMING THAT ONE CAN FACTOR EFFICIENTLY THE ABOVE TASK CAN BE DONE EFFICIENTLY.

THE ALGORITHM USES A CONVEX INTEGER PROGRAM IN FIXED DIMENSION (2 AND 4) WHICH IS IN $\mathcal{P}$ (LENSTRA) AND ALSO SCHOOF'S ALGORITHM.

THE LAST STEP IN THE ALGORITHM INVOLVES FACTORING AN ELEMENT

$$\gamma \in \Gamma = \langle C, T \rangle$$

INTO A WORD WITH MINIMAL T-COUNT.

THE KEY POINT IS THAT THESE SUPER GATES ARE SET UP SO THAT THERE IS AN EXPLICIT HOMOMORPHISM

$$\Gamma \longrightarrow PGL(2, \mathbb{Q}_p)$$

$(p = |C| - 1)$ AND SUCH THAT Γ

Γ ACTS SIMPLY TRANSITIVELY ON THE EDGES OF THE $|C|$-REGULAR TREE

$$X = PGL(2, \mathbb{Q}_p) / PGL(2, \mathbb{Z}_p).$$

THE T-COUNT CORRESPONDING TO DISTANCE MOVED ON THE TREE.

THE MIRACLE OF THESE GATES IS THIS SIMPLE TRANSITIVE ACTION AND THE ARE ONLY FINITELY MANY SUCH Γ'S.