What is...Tropical Geometry?

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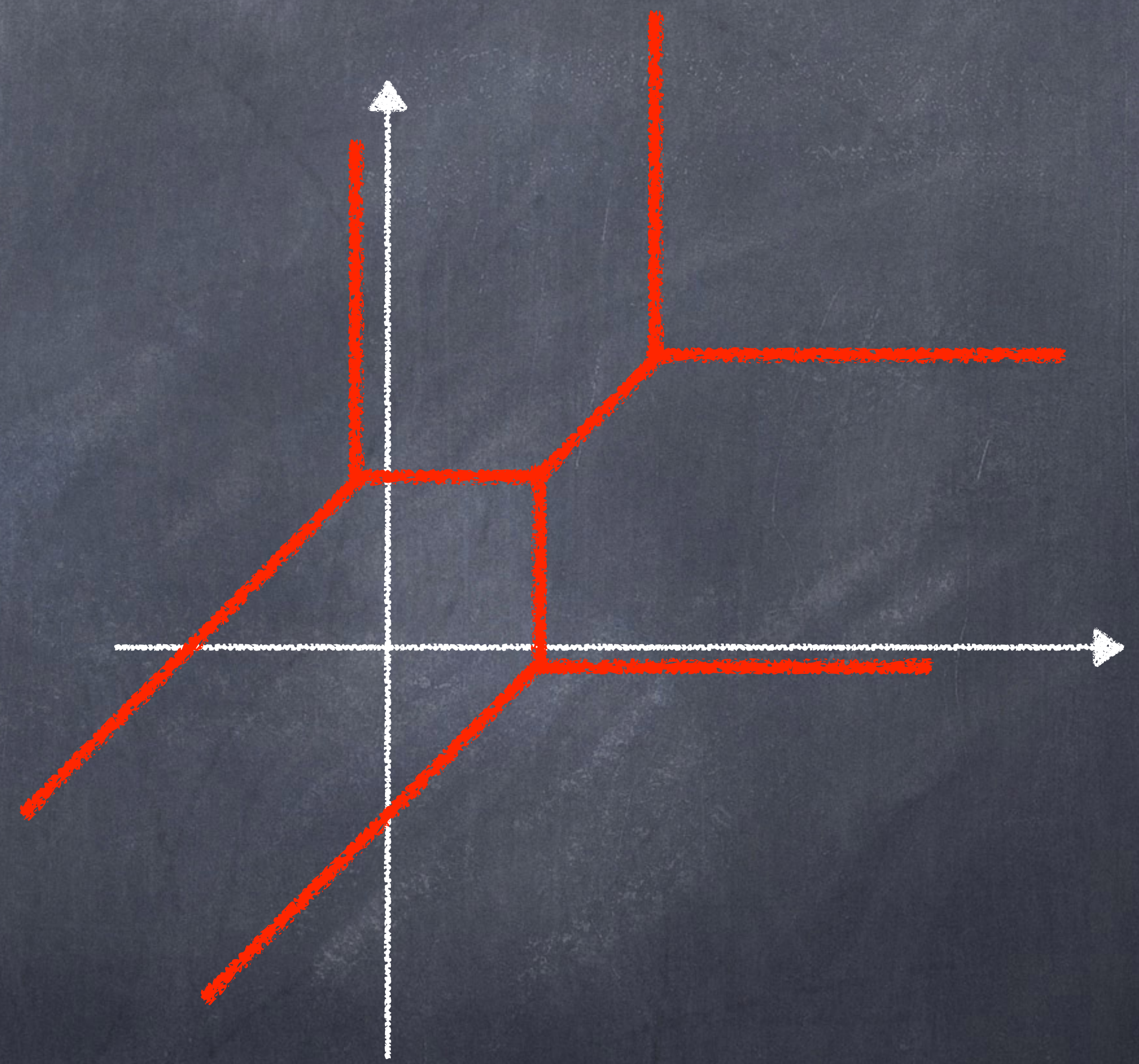
Why tropical?

$5 \odot x \oplus 3 \odot y \oplus 5$

A tropical polynomial



A tropical conic

Why tropical?

This branch of mathematics is called tropical in honor of the Brazilian computer scientist Imre Simon. He lived in São Paulo and commuted across the tropic of Capricorn.

The basics

The fundamental operations are

- **minimum** (the tropical "+") denoted by $\odot$
- **addition** (the tropical "x") denoted by $\oplus$

The tropical numbers are $\mathbf{R} \cup \{\infty\}$.

The basics

EXAMPLE:

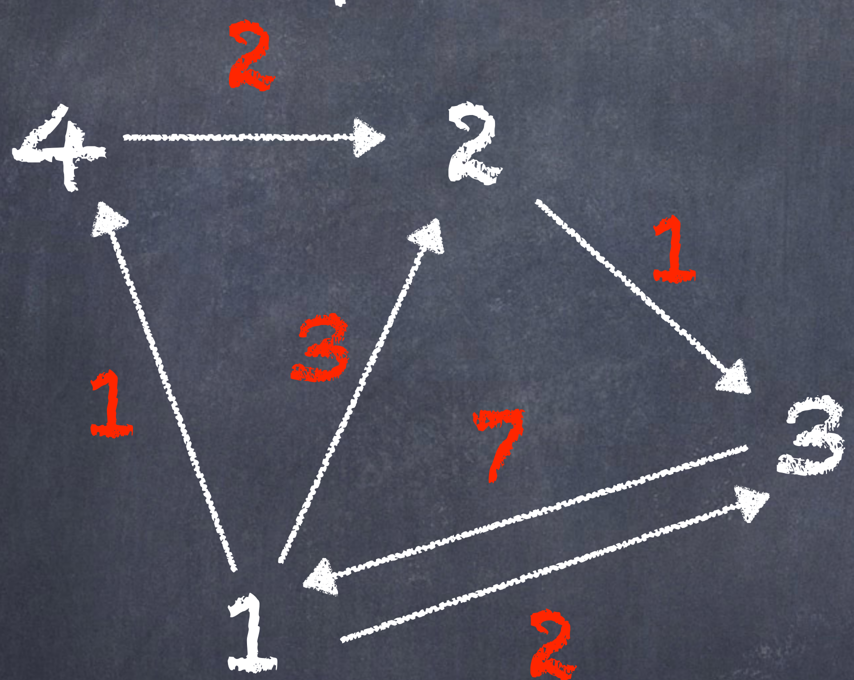
$2 \oplus 3 = 2$ $16 \odot 4 = 20$ $2 \odot (4 \oplus 23) = 6$ $3 \oplus \infty = 3$

MATRIX MULTIPLICATION:

$$\begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix}$$

Optimizations problems

Given a weighted graph find the shortest path between two vertices.



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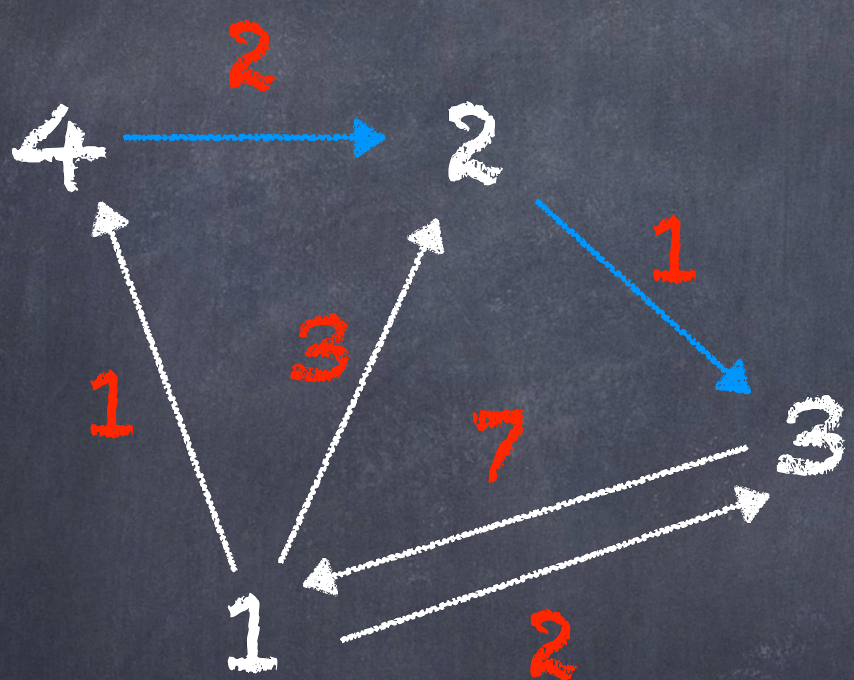
$$\Longleftrightarrow \quad \begin{pmatrix} 0 & 3 & 2 & 1 \\ \infty & 0 & 1 & \infty \\ 7 & \infty & 0 & \infty \\ \infty & 2 & \infty & 0 \end{pmatrix}$$

A(G)

Compute A(G)^3

Optimizations problems

$$A(G)^\wedge 3 = \begin{pmatrix} 0 & 3 & 2 & 1 \\ 8 & 0 & 1 & 9 \\ 7 & 10 & 0 & 8 \\ 10 & 2 & 3 & 0 \end{pmatrix}$$



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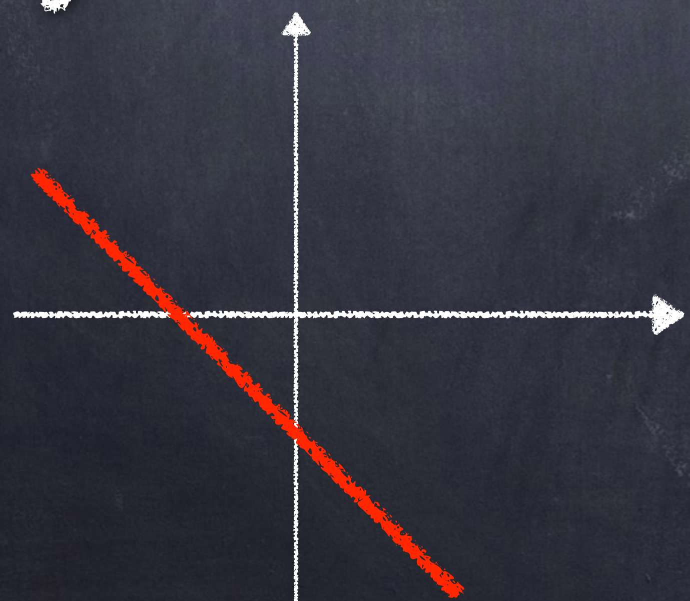
The shortest path between 4 and 3 is 4->2->3 and has length 3

Tropical geometry

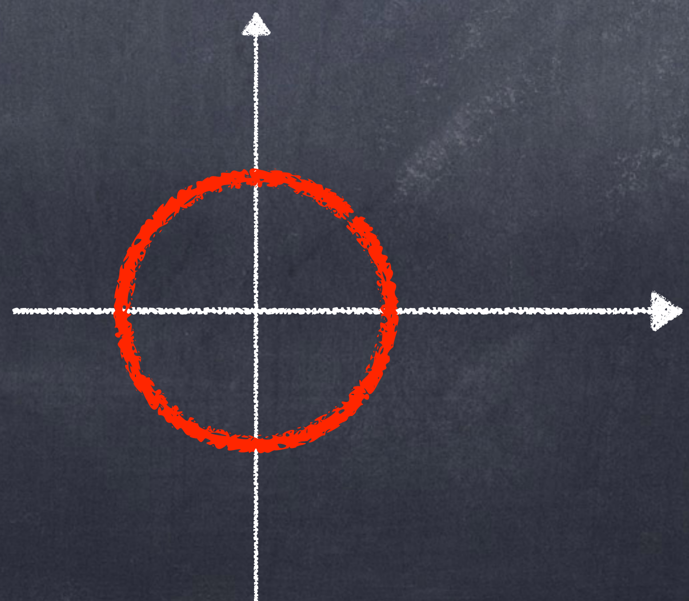
Algebraic geometry is the study of the solutions of systems of polynomial equations in many variables. The set of solution is called an algebraic variety.

EXAMPLE:

x+y+2=0

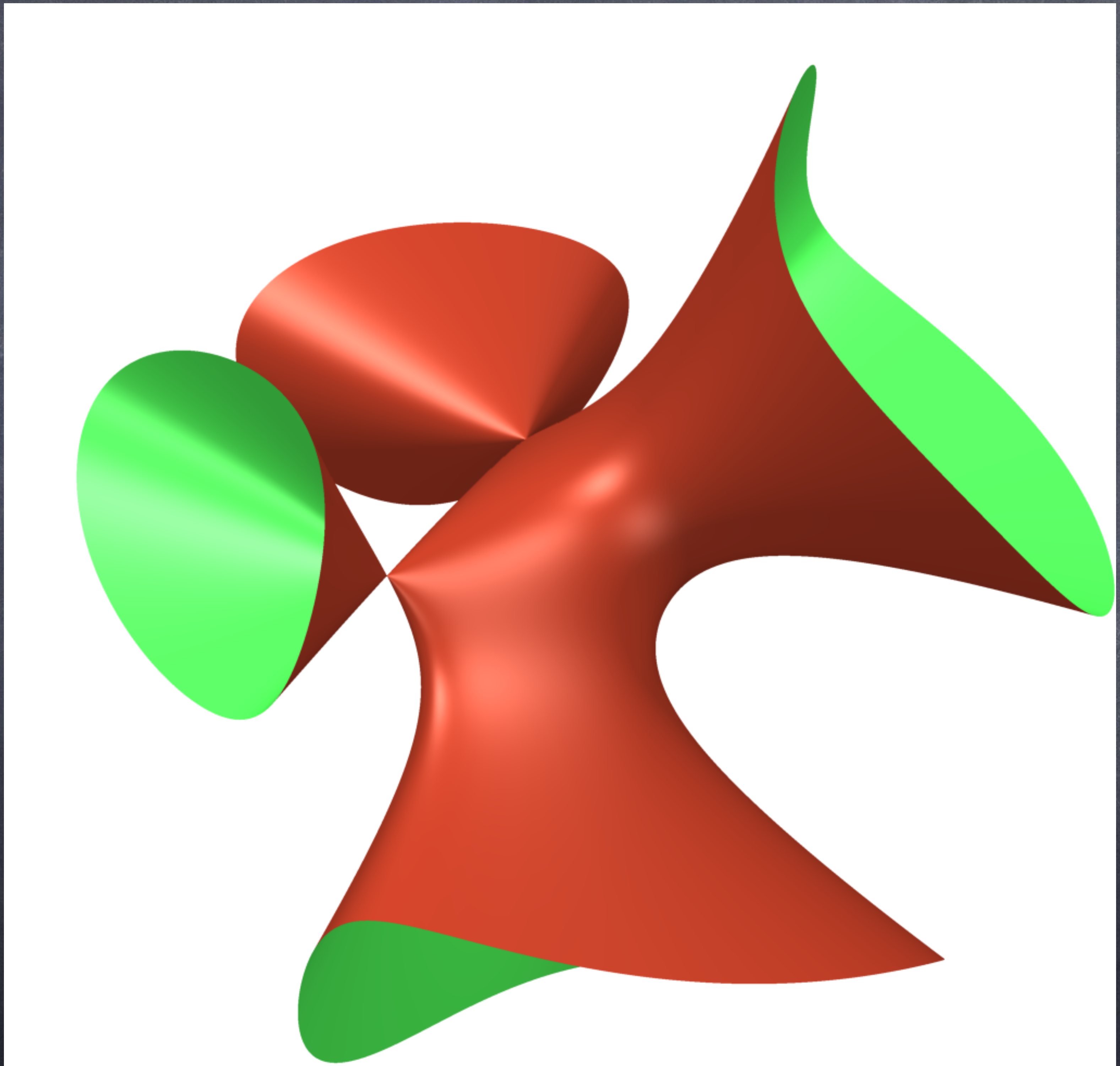


x^2+y^2-1=0



EXAMPLE:

x^2y+y^2+z^2+2xyz-1=0



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Tropical geometry

It is possible to tropicalize polynomials:

$\text{tropicalization}(x+y+2) = x \oplus y \oplus 2 = \min\{x,y,2\}$

$\text{tropicalization}(x^2+y^2+1) = x \odot x \oplus y \odot y \oplus 1 = \min\{2x,2y,1\}$

How to define a tropical variety?

EXAMPLE

$$x^3+6x^2+11x+6=0$$

$$x^3+6x^2+11x+6=(x+1)(x+2)(x+3)$$

$$\downarrow$$

$$x=1, x=2, x=3$$

Tropical geometry

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How to define a tropical variety?

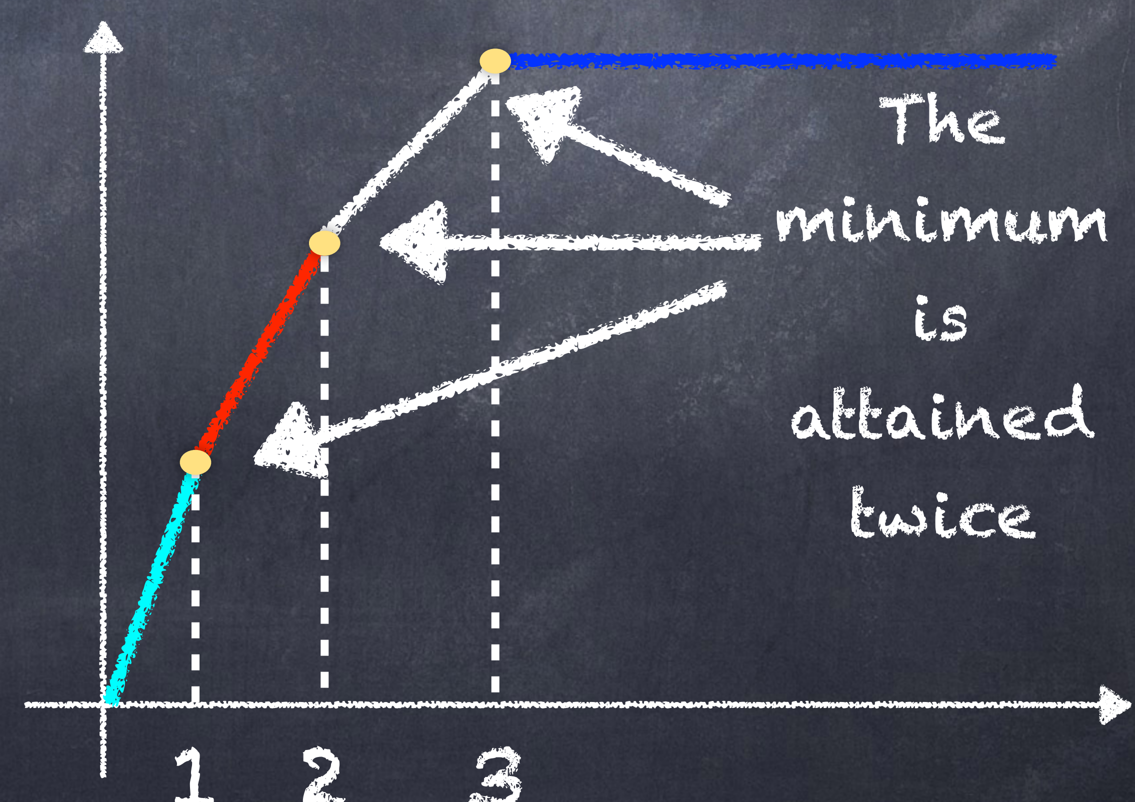
EXAMPLE

$x^3 \oplus 1 \odot x^2 \oplus 3 \odot x \oplus 6 =$

$\min\{3x, 1+2x, 3+x, 6\} =$

$x \odot x \odot x \oplus x \odot x \oplus 3 \odot x \oplus 6 =$

$(x \oplus 1) \odot (x \oplus 2) \odot (x \oplus 3)$



Tropical geometry

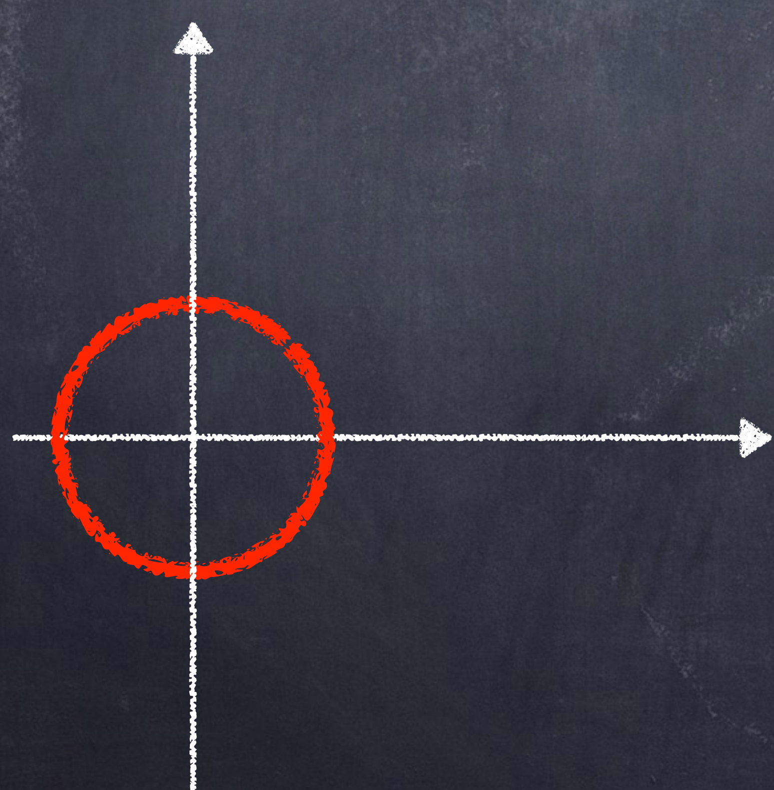
The tropical variety associated to a tropical polynomial trop f is defined by:

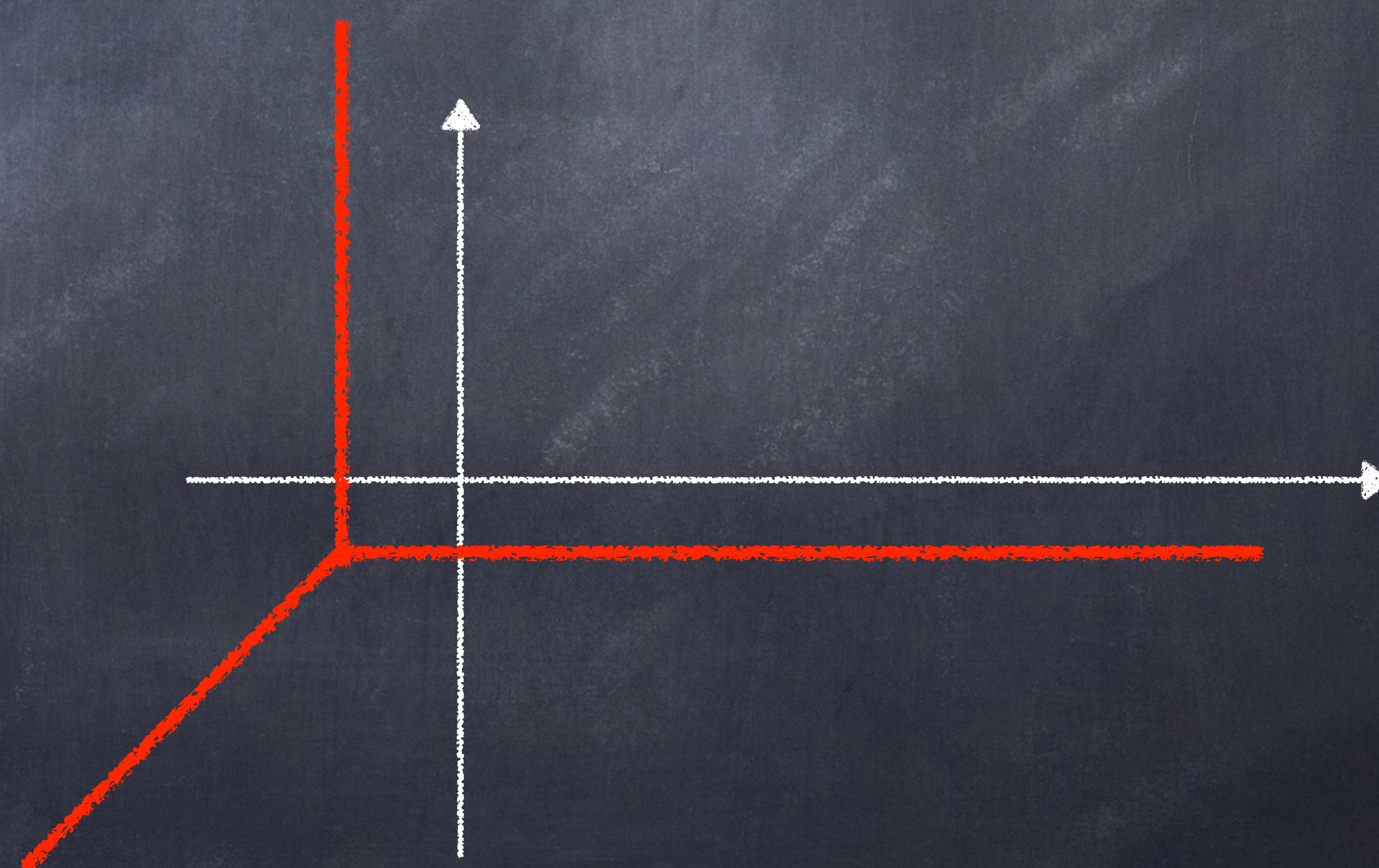
V(trop f)={$(x_1, x_2, \ldots, x_n)$: the minimum in trop f is attained at least twice}

V(x^2⊕y^2⊕-1) ⟹

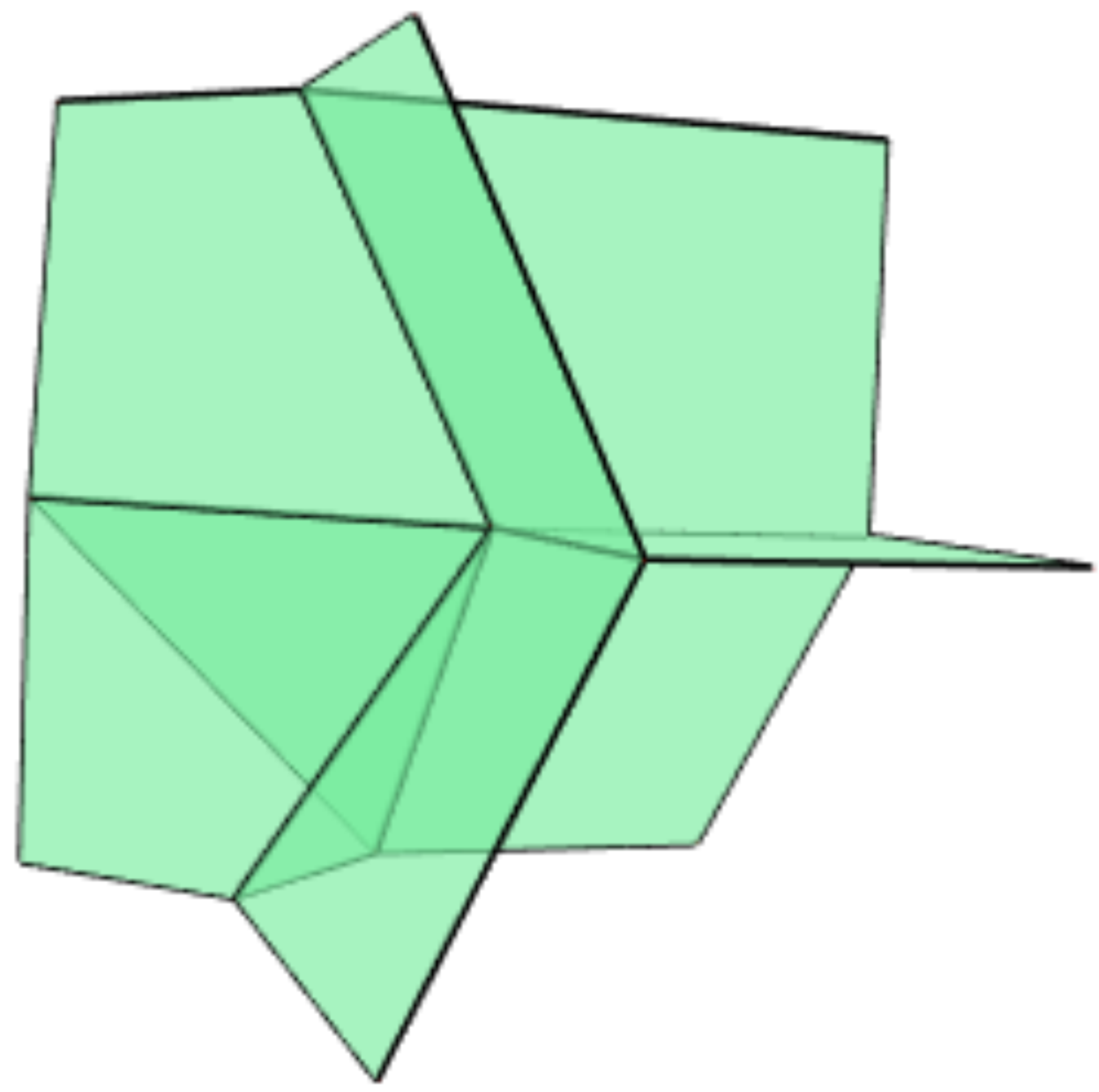
Tropical geometry

Algebraic Varieties → Tropicalization → Tropical Varieties





V(trop (x^2y+y^2+z^2+2xyz-1))



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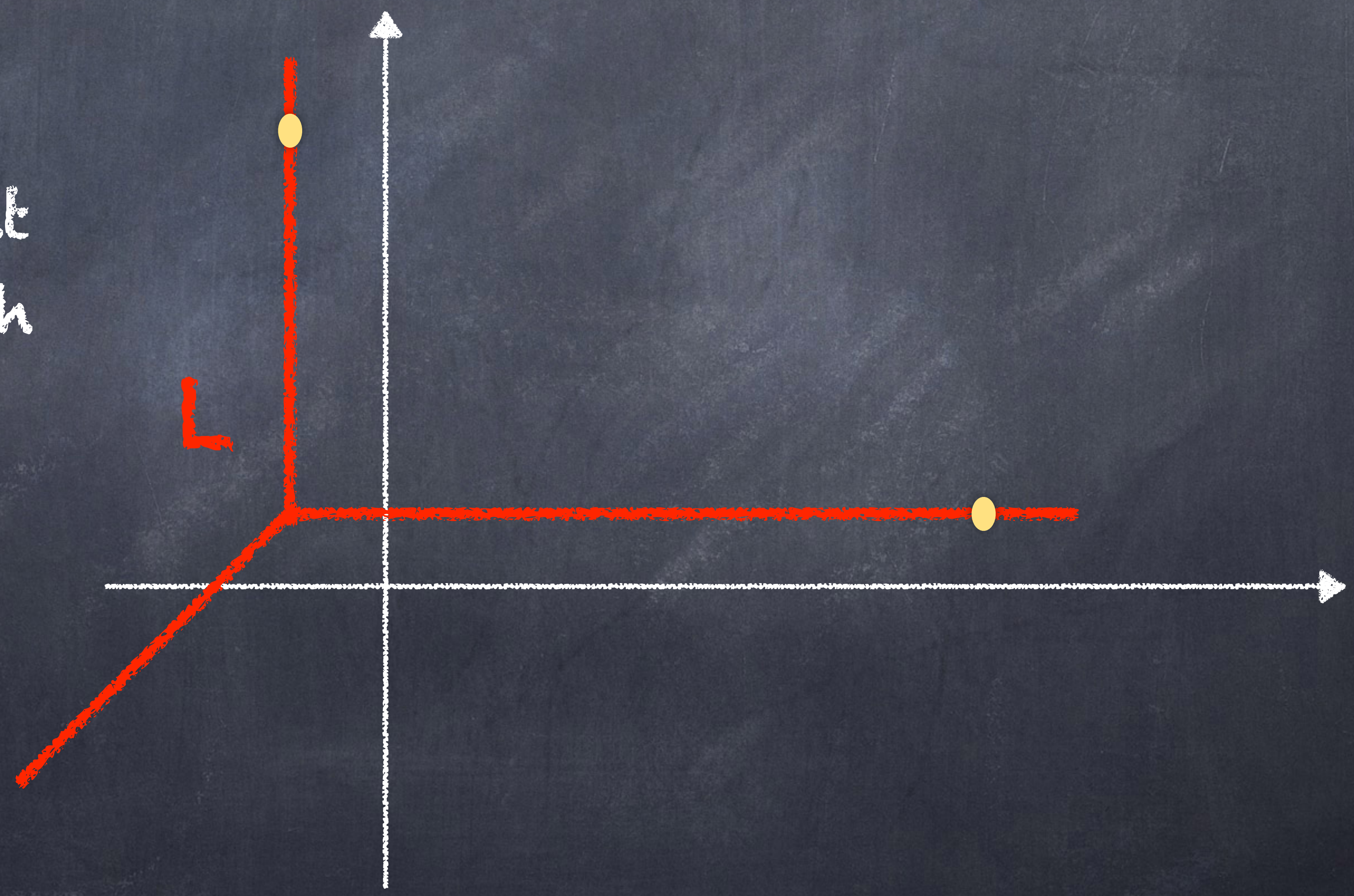
Tropical geometry

Some properties of algebraic varieties hold in the tropical world too.

Given two points there is one straight line passing through them.



$L = V(a \odot x \oplus b \odot y \oplus c)$

The first major result in tropical geometry

Theorem (Mikhalkin 2005)
The number N(d) of rational curves in the plane of degree d passing through 3d-1 points in general position is equal to number of the tropical curves of degree d passing through those points counted with multiplicities.

d=1 lines through 2 points N(1)=1

d=2 conics through 5 points N(2)=1

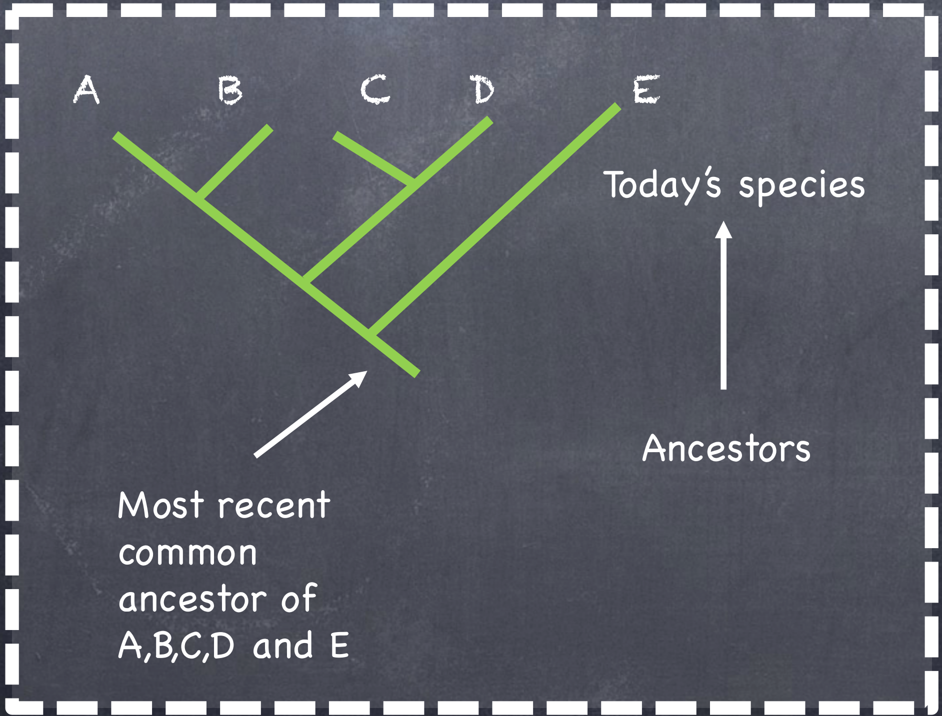
d=3 cubics through 8 points N(3)= 12
(Steiner 1848)

d=4 quartics through 11 points N(4) = 620
(Zeuthen 1873)

Kontsevich in 1992 gives a formula for N(d) for any d.

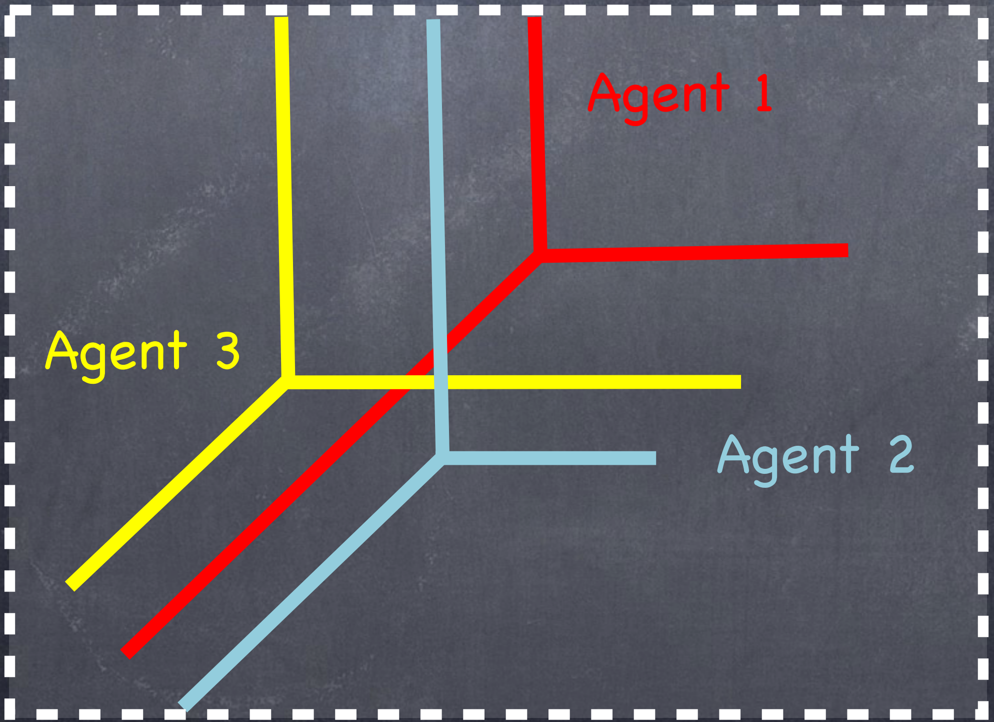
Applications

- Phylogenetics: it is a branch of Biology which studies the evolution of species. The key tool is the phylogenetic tree which records the distance between species. Speyer and Sturmfels proved that the properties that characterize phylogenetic trees are the same that define tropical lines.



Applications

- Economics (Auction Theory): E.Baldwin and P.Klemperer used tropical geometry to understand product-mix auctions in Economics. They represented the different agents in the auction by tropical curves and find the equilibrium by looking at the points of intersection



Summary

- Origins of tropical arithmetics in computer science
- Tropical geometry as tropical algebraic geometry
- Applications
- There is many more directions to explore…

References:

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