What is ... Harmonic Measure?

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Introduction

We'll discuss harmonic measure from 2 points of view, both related to heat conduction:

1. **Physical/Probabilistic**
2. **Solution of Boundary Value Problems**

Physical/Probabilistic Interpretation of Harmonic Measure

- **Heat Conduction**. Heat is conducted through a fluid by the transfer of kinetic energy from molecule to molecule.

Physical/Probabilistic Interpretation of Harmonic Measure

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- **Brownian Motion**. Physically, the random motion of a particle suspended in a fluid, due to buffeting by the molecular collisions. (Theory developed by Einstein in 1905).

Physical/Probabilistic Interpretation of Harmonic Measure

Let Ω be a connected open region (aka, a **domain**) in Euclidean space (in 2-D or 3-D, to keep things simple), with boundary (i.e., perimeter) $\partial\Omega$.

Consider a Brownian traveler *starting* from some point $P \in \Omega$, and eventually reaching the boundary $\partial\Omega$.

Let's momentarily fix this starting point P.

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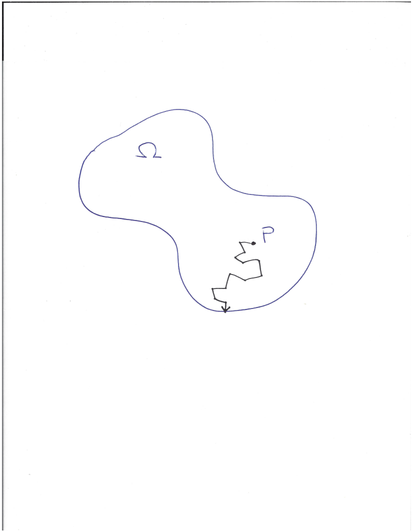
- **Harmonic Measure**. For a subset $A \subset \partial\Omega$, the **harmonic measure** of A, denoted $\omega^P(A)$, is the probability that a Brownian traveler's first contact with $\partial\Omega$ is somewhere in the set A.

Physical/Probabilistic Interpretation of Harmonic Measure



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- Harmonic measure is really a *family* of measures (a different one for each starting point P).
- Harmonic measure is a *probability measure*; i.e., the total mass $\omega^P(\partial\Omega) = 1$; i.e., every Brownian traveler from any given starting point P will eventually reach the boundary with probability 1.

Remarks (cont.).

- There are other (standard) ways to measure sets on the boundary: arclength in 2-D (the boundary is a curve), surface area in 3-D (the boundary is a surface). We denote these by σ, i.e., if A is some piece of the boundary, then $\sigma(A)$ is the length of A (in 2-D), or the surface area of A (in 3-D).

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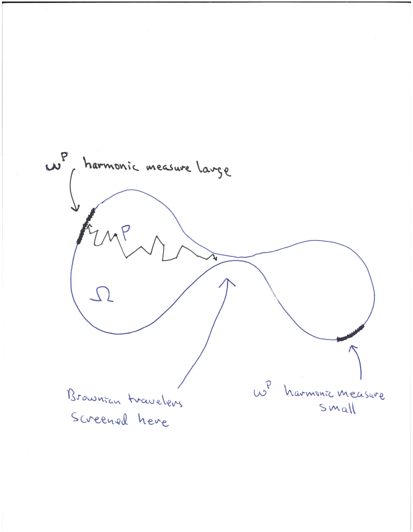
- The harmonic measure size of a piece of the boundary depends on the shape of Ω, *relative* to the starting point P; i.e., $\omega^P(A)$ will be small if P is screened from A by some other part of the boundary, while $\omega^P(A)$ will be large if Brownian travelers from P have easy access to A.

Physical/Probabilistic Interpretation of Harmonic Measure



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- ...also, vice versa...?

Not surprisingly, the answer to these questions is tied to the geometry of Ω and its boundary.

Boundary Value Problem Interpretation of Harmonic Measure

Harmonic measure has another significance, which we again relate to heat conduction.

Consider a domain Ω as above, which is in a thermal steady-state, i.e., the temperature at each point in Ω does not change in time (although it may vary from one point to another).

For each point P in Ω, let $u(P)$ denote the temperature at the point P.

Boundary Value Problem Interpretation of Harmonic Measure

For simplicity, let's look at the 2-D case. We may then write $P = (x, y)$, the usual coordinates in the Cartesian x-y plane.

The temperature function u satisfies the **Steady-State Heat Equation**, aka **Laplace's Equation**, which in 2-D is as follows:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

I.e., we take the second order derivative in each of the x and y directions (and in the z direction in 3-D), and then add them up.

Boundary Value Problem Interpretation of Harmonic Measure

Fundamental question: Suppose we know the temperature at each point on the boundary $\partial\Omega$. We let $f(Q)$ denote the (known) temperature at each point $Q \in \partial\Omega$. Does this determine the temperature function *inside* Ω? And can we compute the temperatures inside, from the boundary temperatures?

I.e., can we find the temperature function u inside Ω, provided that we know that $u = f$ on the boundary?

Boundary Value Problem Interpretation of Harmonic Measure

I.e., we are asking if we can solve the following "Boundary Value Problem":

$$(BVP) \quad \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

Boundary Value Problem Interpretation of Harmonic Measure

Answer: if $\partial\Omega$ is not too terrible (and it can actually be fairly wild) then we can do this using the harmonic measure formula:

$$u(P) = \int_{\partial\Omega} f(Q)\, d\omega^P(Q)\,. \tag{1}$$

What does this mean? Let's give an intuitive explanation by analogy to integration in first year calculus.

Boundary Value Problem Interpretation of Harmonic Measure

Recall that ω^P measures (in a certain sense), the size of sets on $\partial\Omega$. In 1-D calculus, we measure the size of an interval [a,b] by simply computing its length, which we can express as an integral:

$$\ell[a,b] = b - a = \int_a^b 1\,dx\,.$$

I.e., dx corresponds to standard 1-D size (the length); analogously, integration with respect to $d\omega^P$ corresponds to the harmonic measure size.

Boundary Value Problem Interpretation of Harmonic Measure

More generally, in place of $\int_a^b 1\,dx$, we can consider

$$\int_a^b f(x)\,dx\,,$$

for any (let's say continuous) function $f(x)$.

The integral in formula (1) above (repeated here):

$$u(P) = \int_{\partial\Omega} f(Q)\,d\omega^P(Q)$$

is again analogous to this.

Thank you!